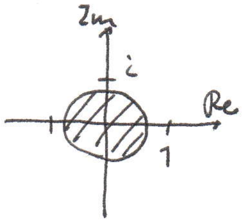
物理情報数学A - 問題解答

34.

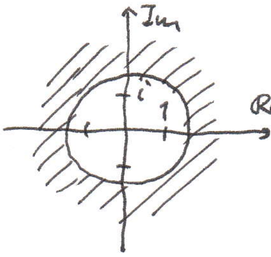
(1) $f(z) = \dfrac{-1}{1-z} = -\displaystyle\sum_{n=0}^{\infty} z^n$ 〜 $|z|<1$ で収束 (テーラー展開)

‖
べき級数展開.



(2) $|z|>1 \Leftrightarrow \left|\dfrac{1}{z}\right| < 1$ より.

$$f(z) = \frac{\frac{1}{z}}{1-\frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$ (ローラン展開)

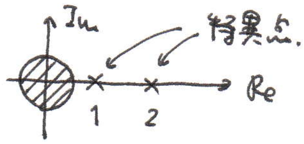


(3) $|z|>2$ より. $\left|\dfrac{1}{z}\right| < \dfrac{1}{2} < 1$ ゆえに. (2)と同じローラン展開が可能.

35.

$$f(z) = \frac{z}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{2}{z-2}$$

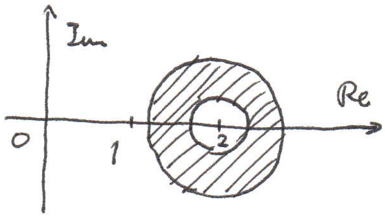
(1)



(1) $|z|<1$ で $f(z)$は正則. $|z|<1$, $\left|\dfrac{z}{2}\right| < \dfrac{1}{2} < 1$ より. 次のテーラー展開が可能:

$$\therefore\ f(z) = \frac{1}{1-z} + \frac{-1}{1-\left(\frac{z}{2}\right)} = \sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty}\left(1-\frac{1}{2^n}\right)z^n$$

(2) $$\frac{-1}{z-1} = \frac{-1}{1+(z-2)} = \frac{-1}{1-\left(\frac{z-2}{-1}\right)}$$



ここで, $0 < |z-2| < 1$ より. $\left|\dfrac{z-2}{-1}\right| < 1$.

$$\therefore\ \frac{-1}{z-1} = -\sum_{n=0}^{\infty}\left(\frac{z-2}{-1}\right)^n = \sum_{n=0}^{\infty}(-1)^{n-1}(z-2)^n$$

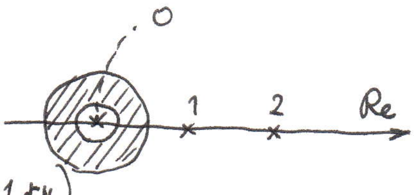
$$\therefore\ f(z) = \frac{2}{z-2} + \sum_{n=0}^{\infty}(-1)^{n-1}(z-2)^n$$ (ローラン展開)

36.

$$f(z) = \frac{2}{z(z-1)(z-2)} = \frac{1}{z} + \frac{-2}{z-1} + \frac{1}{z-2}$$ と分解可能.

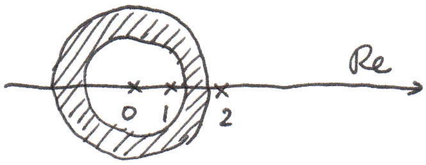
(1) $$\frac{-2}{z-1} = \frac{2}{1-z} = 2\sum_{n=0}^{\infty} z^n$$ ($|z|<1$ より)

$$\frac{1}{z-2} = \frac{-1}{2-z} = \frac{-\frac{1}{2}}{1-\left(\frac{z}{2}\right)} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$ ($\left|\frac{z}{2}\right|<1$ より)



$$\therefore\ f(z) = \frac{1}{z} + 2\sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} = \frac{1}{z} + \sum_{n=0}^{\infty}\left(2-\frac{1}{2^{n+1}}\right)z^n$$

(2) $\dfrac{2}{1-z} = \dfrac{\frac{2}{z}}{\frac{1}{z}-1} = -\dfrac{2}{z}\cdot\dfrac{1}{1-\left(\frac{1}{z}\right)}$



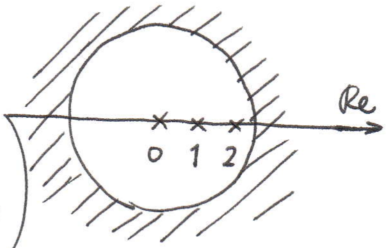
$= -\dfrac{2}{z}\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \quad \left(\because \left|\frac{1}{z}\right| = \frac{1}{|z|} < 1 \text{ より}\right)$

$\dfrac{1}{z-2} = \dfrac{-1}{2-z} = \dfrac{-\frac{1}{2}}{1-\left(\frac{z}{2}\right)} = -\dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \quad \left(\because \left|\frac{z}{2}\right| = \frac{|z|}{2} < \frac{2}{2} = 1 \text{ より}\right)$

$\therefore f(z) = \dfrac{1}{z} - 2\displaystyle\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}z^n = -2\sum_{n=1}^{\infty}\frac{1}{z^{n+1}} - \frac{1}{z} - \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}z^n$

(3) $\dfrac{-2}{z-1} = \dfrac{-\frac{2}{z}}{1-\frac{1}{z}} = -\dfrac{2}{z}\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = -2\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$

$\left(\because \left|\dfrac{1}{z}\right| = \dfrac{1}{|z|} < \dfrac{2}{|z|} < 1\right)$



$\dfrac{1}{z-2} = \dfrac{\frac{1}{z}}{1-\frac{2}{z}} = \dfrac{1}{z}\displaystyle\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$

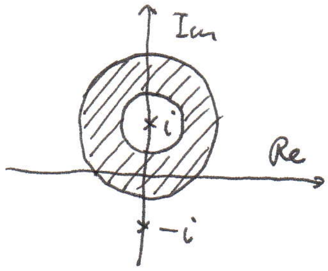
$\left(\because \left|\dfrac{2}{z}\right| < \dfrac{2}{|z|} < 1\right)$

$\therefore f(z) = \dfrac{1}{z} - 2\displaystyle\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}2^n\cdot\frac{1}{z^{n+1}} = \frac{1}{z} + \sum_{n=0}^{\infty}(2^n-2)\frac{1}{z^{n+1}} = \sum_{n=2}^{\infty}(2^n-2)\frac{1}{z^{n+1}}$

37.

$f(z) = \dfrac{1}{1+z^2} = \dfrac{1}{2i}\left(\dfrac{1}{z-i} - \underbrace{\dfrac{1}{z+i}}\right)$

↳ここを展開.



$\dfrac{1}{z+i} = \dfrac{-1}{-2i-(z-i)} = \dfrac{\frac{1}{2i}}{1-\left(\frac{z-i}{-2i}\right)}$

ここで、$\left|\dfrac{z-i}{-2i}\right| = \dfrac{|z-i|}{2} < 1$ より. $\dfrac{1}{z+i} = \dfrac{1}{2i}\displaystyle\sum_{n=0}^{\infty}\left(\frac{z-i}{-2i}\right)^n$

$\therefore f(z) = \dfrac{1}{2i}\cdot\dfrac{1}{z-i} - \dfrac{1}{2i}\cdot\dfrac{1}{2i}\displaystyle\sum_{n=0}^{\infty}\frac{1}{(-2i)^n}(z-i)^n = \frac{1}{2i}\cdot\frac{1}{z-i} - \sum_{n=0}^{\infty}\frac{1}{(-2i)^{n+2}}(z-i)^n$

38.

(1). $$f(z) = \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots}{z^2\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)} = \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots}{z^3\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)}$$

$$= \frac{1}{z^3}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right)\left(1 + \frac{z^2}{3!} - \frac{z^4}{5!} - \cdots\right)$$

$$\frac{1}{1+w} = \sum_{n=0}^{\infty}(-w)^n = 1 - w + w^2 - \cdots$$

$(|w|$：十分に小さい$)$

これで $w = -\frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$ とした．

$$= \frac{1}{z^3}\left(1 + \frac{z^2}{3!} - \frac{z^2}{2!} + (z^4\text{より高次})\right)$$

$$= \frac{1}{z^3} - \frac{1}{3}\cdot\frac{1}{z} + (z\text{より高次})$$

ゆえに，$f(z)$ の主要部 $= \frac{1}{z^3} - \frac{1}{3}\cdot\frac{1}{z}$

(2). $w = z - 1$ とおくて，

$$f(z) = f(w) = \frac{(w+1)^2}{w^3(w+2)^3} = \frac{1}{w^3}(w^2 + 2w + 1)\left(\frac{1}{2} - \frac{w}{4} + \frac{w^2}{8} + (w^3\text{より高次})\right)^3$$

$$\frac{1}{w+2} = \frac{\frac{1}{2}}{1 - \left(-\frac{w}{2}\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{w}{2}\right)^n$$

$$= \frac{1}{2} - \frac{w}{4} + \frac{w^2}{8} + \cdots \quad (|w|：\text{十分に小さい})$$

$$= \frac{1}{w^3}(w^2 + 2w + 1)\left(\frac{1}{8} - \frac{3}{16}w + \frac{3}{16}w^2 + (w^3\text{より高次})\right)$$

$$= \frac{1}{8}w^{-3} + \frac{1}{16}w^{-2} - \frac{1}{16}w^{-1} + (\text{定数項より高次})$$

ゆえに，$f(z)$ の主要部 $= \frac{1}{8}\cdot\frac{1}{(z-1)^3} + \frac{1}{16}\cdot\frac{1}{(z-1)^2} - \frac{1}{16}\cdot\frac{1}{(z-1)}$