物理情報数学A - 問題解答

25. (1) C上の点 $(x(t), y(t))$ は, $x(t) = \cos t,\ y(t) = \sin t$ $(t : 0 \to \pi)$ とおける.

$$\therefore \int_C dx = \int_0^\pi \frac{dx(t)}{dt}\,dt = \int_0^\pi -\sin t\,dt = \cos t\Big|_0^\pi = -2$$

$$\int_C dy = \int_0^\pi \frac{dy(t)}{dt}\,dt = \int_0^\pi \cos t\,dt = \sin t\Big|_0^\pi = 0.$$

(2)

$$\int_C x\,dx = \int_0^\pi x(t)\cdot\frac{dx(t)}{dt}\,dt = \int_0^\pi \cos t\cdot(-\sin t)\,dt = -\frac{1}{2}\int_0^\pi \sin 2t\,dt$$

$$= -\frac{1}{2}\cdot\frac{-1}{2}\cos 2t\Big|_0^\pi = 0$$

$$\int_C x\,dy = \int_0^\pi x(t)\cdot\frac{dy(t)}{dt}\,dt = \int_0^\pi \cos t\cdot\cos t\,dt = \frac{1}{2}\int_0^\pi (1+\cos 2t)\,dt$$

$$= \frac{1}{2}\left(t + \frac{1}{2}\sin 2t\right)\Big|_0^\pi = \frac{\pi}{2}$$

26. (1)

$$x = \frac{z+\bar{z}}{2},\quad y = \frac{z-\bar{z}}{2i} \text{ より } \frac{\partial x}{\partial \bar{z}} = \frac{1}{2},\quad \frac{\partial y}{\partial \bar{z}} = -\frac{1}{2i} = \frac{i}{2}$$

$$\therefore \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y}$$

(2)

$$\oint_C f(z)\,dz = \int_D \left[-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} - i\frac{\partial v}{\partial y} + i\frac{\partial u}{\partial x}\right]dxdy$$

$$= \int_D \left[i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) - \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right]dxdy$$

$$= \int_D \left[i\frac{\partial}{\partial x}(u+iv) - \frac{\partial}{\partial y}(u+iv)\right]dxdy = \int_D \left(i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right)dxdy$$

$$= 2i\int_D \left(\frac{1}{2}\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y}\right)dxdy = 2i\int_D \frac{\partial f}{\partial \bar{z}}\,dxdy$$

(3) (2) で $f(z) = \bar{z}$ とすると, $\oint_C \bar{z}\,dz = 2i\int_D dxdy.$

$$\therefore (\text{Dの面積}) = \int_D dxdy = \frac{1}{2i}\oint_C \bar{z}\,dz.$$