物理情報数学A - 問題解答

13. (1) $$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}\frac{\overline{z+h}-\bar{z}}{h} = \lim_{h\to 0}\frac{\bar{h}}{h}$$

$h=re^{i\theta}$ とおくと, $$= \lim_{r\to 0}\frac{re^{-i\theta}}{re^{i\theta}} = e^{-2i\theta}$$ これは θ に依存するので $f(z)$ は微分不可能.

(2) $$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}\frac{(\overline{z+h})^2-\bar{z}^2}{h} = \lim_{h\to 0}\frac{(\bar{z}+\bar{h})^2-\bar{z}^2}{h}$$

$$= \lim_{h\to 0}\frac{2\bar{z}\bar{h}+\bar{h}^2}{h} = \lim_{r\to 0}\frac{2\bar{z}\cdot re^{-i\theta}+(re^{-i\theta})^2}{re^{i\theta}} = 2\bar{z}e^{-2i\theta}$$

$h=re^{i\theta}$ とおいた. これは θ に依存するので, $f(z)$ は微分不可能.

(3) $$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}\frac{e^{z+h}-e^z}{h} = \lim_{h\to 0}\frac{e^z}{h}(e^h-1)$$

$$= \lim_{h\to 0}\frac{e^z}{h}\left(1+h+\frac{h^2}{2!}+\frac{h^3}{3!}+\cdots-1\right)$$ （e^z は $\sum_{n=0}^{\infty}\frac{z^n}{n!}$ で定義される）

$$= \lim_{h\to 0}\frac{e^z}{h}\left(h+\frac{h^2}{2!}+\frac{h^3}{3!}+\cdots\right) = \lim_{h\to 0}e^z\left(1+\frac{h}{2!}+\frac{h^3}{3!}+\cdots\right) = e^z$$

ゆえに $f(z)=e^z$ はすべての z で微分可能.

14. (1) $f(z)=f(x+iy)=(x+iy)^2 = x^2-y^2+2ixy$. $\therefore$ $u(x,y)=x^2-y^2$, $v(x,y)=2xy$.

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y} = 2x-2x=0,\quad \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y} = 2y-2y=0.$$

ゆえに, C-R関係式が成立するので, $f(z)=z^2$ はすべての z で微分可能. また,

$$\frac{df(z)}{dz} = \frac{\partial}{\partial x}f(x+iy) = 2x+2iy = 2(x+iy) = 2z.$$

(2) $f(z)=f(x+iy)=(\overline{x+iy})^2=(x-iy)^2 = x^2-y^2-2ixy$. $\therefore \begin{cases} u = x^2-y^2 \\ v=-2xy \end{cases}$

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y} = 2x+2x=4x,\quad \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y} = -2y-2y=-4y.$$

ゆえに, $(x,y)=(0,0)$ を除くすべての z で $f(z)=\bar{z}^2$ は微分不可能.

(3) $f(z)=f(x+iy)=e^{x+iy} = e^x(\cos y+i\sin y)$ $\therefore$ $u=e^x\cos y$, $v=e^x\sin y$.

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y} = e^x\cos y-e^x\cos y=0,\quad \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y} = e^x\sin y-e^x\sin y=0.$$

ゆえに $f(z)$ はすべての z で微分可能で, $\frac{df}{dz} = \frac{\partial}{\partial x}f(x+iy) = e^x(\cos y+i\sin y) = e^z$

i.e., $\frac{d}{dz}e^z = e^z$.

15. (1) $f(z) = f(x+iy) = e^{i(x+iy)} = e^{-y}e^{ix} = e^{-y}(\cos x + i\sin x)$

$\therefore\ u = e^{-y}\cos x,\quad v = e^{-y}\sin x$

$\to\ \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = -e^{-y}\sin x - (-1)\cdot e^{-y}\sin x = 0,$

$\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} = e^{-y}\cos x - e^{-y}\cos x = 0.$ 故に $f(z) = e^{iz}$ はすべての z で微分可能。

このとき、$\dfrac{df(z)}{dz} = \dfrac{\partial}{\partial x}f(x+iy) = e^{-y}(-\sin x + i\cos x)$

$= e^{-y}\cdot i(\cos x + i\sin x) = ie^{iz}$

次に $f(z) = e^{-iz}$ については、$f(x+iy) = e^{-i(x+iy)} = e^{y}(\cos x - i\sin x)$ より、

$\begin{cases} u = e^{y}\cos x \\ v = -e^{y}\sin x \end{cases}$

$\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = -e^{y}\sin x - (-e^{y}\sin x) = 0,$

$\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} = -e^{y}\cos x + e^{y}\cos x = 0.$

故に $f(z) = e^{-iz}$ はすべての z で微分可能で、$\dfrac{df(z)}{dz} = \dfrac{\partial}{\partial x}f(x+iy) = e^{y}(-\sin x - i\cos x)$

$= -ie^{y}(\cos x - i\sin x) = -ie^{-iz}$

(2) 微分演算の線形性 $\left(\dfrac{d}{dz}(f+g) = \dfrac{df}{dz} + \dfrac{dg}{dz}\right)$ より、

$\dfrac{d}{dz}\sin z = \dfrac{d}{dz}\cdot\dfrac{e^{iz} - e^{-iz}}{2i} = \dfrac{1}{2i}\cdot ie^{iz} - \dfrac{1}{2i}(-i)e^{-iz} = \dfrac{e^{iz} + e^{-iz}}{2} = \cos z$

$\dfrac{d}{dz}\cos z = \dfrac{d}{dz}\cdot\dfrac{e^{iz} + e^{-iz}}{2} = \dfrac{1}{2}\cdot ie^{iz} + \dfrac{1}{2}(-i)e^{-iz} = \dfrac{e^{iz} - e^{-iz}}{-2i} = -\sin z$

16. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial v}{\partial y}\right) + \dfrac{\partial}{\partial y}\left(-\dfrac{\partial v}{\partial x}\right)$

$= \dfrac{\partial^2 v}{\partial x\partial y} - \dfrac{\partial^2 v}{\partial x\partial y} = 0.$

$\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial v}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{\partial v}{\partial y}\right) = \dfrac{\partial}{\partial x}\left(-\dfrac{\partial u}{\partial y}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)$

$= -\dfrac{\partial^2 u}{\partial x\partial y} + \dfrac{\partial^2 u}{\partial x\partial y} = 0.$

17. (1) 条件より, $f(z)=u+iv$ は全領域でC-R関係式 $\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=0$, $\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=0$ をみたす。

いま $u=x^2-y^2$ なので これを $\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=0$ に代入すると,

$$2x-\frac{\partial v}{\partial y}=0 \longrightarrow v(x,y)=2xy+g(x)$$

ただし, $g(x)$ は適当な x の関数. これが $\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=0$ をみたすので,

$$-2y+2y+\frac{dg(x)}{dx}=0 \qquad \therefore \frac{d}{dx}g(x)=0$$

ゆえに $g(x)=c$ (定数) より, $v(x,y)=2xy+c$.

このとき $f(z)=x^2-y^2+i(2xy+c)=(x+iy)^2+ic=z^2+ic$

(2) $u=x^n$ を $\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=0$ に代入すると, $\frac{\partial v}{\partial x}=0$. ゆえに $v=h(y)$

と決まる. ($h(y)$ は適当な y の関数). これが $\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=0$ をみたすので

$$nx^{n-1}-\frac{dh(y)}{dy}=0. \longrightarrow nx^{n-1} \text{ は定数でなければならないので,}$$

$n=0$ または $n=1$.

- $n=0$ のとき, $\frac{dh}{dy}=0$ より, $h(y)=c$ (定数). $\therefore f(z)=1+ic$
- $n=1$ のとき, $\frac{dh}{dy}=1$ より, $h(y)=y+c'$ (c'は定数)

$$\therefore f(z)=u+iv=x^1+i(y+c')=(x+iy)+ic'=z+ic'$$

18. (1) $u^2+v^2=K$ の両辺を x で偏微分すると, $2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}=0$.

C-R関係式より $\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$. $\therefore u\frac{\partial u}{\partial x}-v\cdot\frac{\partial u}{\partial y}=0$. —①

次に, yで偏微分した式: $2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}=0$ に $\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}$ を代入すれば,

$u\frac{\partial u}{\partial y}+v\frac{\partial u}{\partial x}=0$. —② . ①, ② を合わせて, 題意の行列方程式を得る.

$K=0$ のとき, 明らかに $u=v=0$. $\therefore f(z)=0$.

(2) $\det\begin{pmatrix} u & -v \\ v & u \end{pmatrix}=u^2+v^2=K$ より, $K>0$ であれば $\begin{pmatrix} u & -v \\ v & u \end{pmatrix}$ には逆行列が

存在する. ゆえに 行列方程式の両辺に $\begin{pmatrix} u & -v \\ v & u \end{pmatrix}^{-1}$ をかければ, $\frac{\partial u}{\partial x}=0$, $\frac{\partial u}{\partial y}=0$

を得る. C-R関係式より, これは $\frac{\partial v}{\partial x}=0$, $\frac{\partial v}{\partial y}=0$ を導く. これより,

$u=u_0$, $v=v_0$ (u_0, v_0は定数). ゆえに $f(z)=u_0+iv_0=$定数.